AD-A234 729

CUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED FINAL 1 Jul88 to 31 Dec 90

4. TITLE AND SUBTITLE ~~STABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS~~ See Title on Report	5. FUNDING NUMBERS AFOSR-88-0256 61102F 2304/A5
6. AUTHOR(S) Mark Pinksy and Volker Wihstutz	AFOSR-TR- 91 0274
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Northwester Univeristy 633 Clark Street Evanston, Il 60208	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	10. SPONSORING/MONITORING AGENCY REPORT NUMBER AFOSR-88-0256
11. SUPPLEMENTARY NOTES	DTIC ELECTE APR 18 1991
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release, distribution unlimited.	12b. DISTRIBUTION CODE C

13. ABSTRACT (*Maximum 200 words*)

In the joint work with Volker Wihstutz of the University of North Carolina, we have investigated the Lyapunov stability of systems defined by a system of differential equations with a stochastic driving term, which may be either white noise or real noise. In the first case we showed that for nilpotent systems it is possible to compute an arbitrary number of terms in the asymptotic expansion of Lyapunov exponent in fractional powers of the noise coefficient when this tends to zero. This includes the important case of critically damped oscillator, which had not been treated previously. These results were then extended to the case of the same nilpotent system driven by a finite-state Markov noise process. This was obtained by a method of homogenization, using techniques previously established to study the central limit theorem for functions of a centered Markov chain. It was shown, as in the case of white noise, that the Lyapunov exponent admits an expansion in fractional powers of the noise parameter, and that the first term of this expansion agrees exactly with the result obtained in the white noise case.

14. SUBJECT TERMS 91 4 16 070	15. NUMBER OF PAGES
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

Lyapunov exponents and rotation numbers of linear systems with real noise

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*Research supported by Air Force Office of Scientific Research

1. Introduction.

A large literature has been devoted to studying the asymptotic properties of the linear stochastic system

$$X_t' = AX_t + \varepsilon BX_tF(\xi_t), \qquad X_0 = x \in R^d \tag{1.1}$$

where A, B are constant $d \times d$ matrices and $F(\xi_t)$ is a mean-zero function of an ergodic Markov process on a compact state space M. Of particular interest is the top Lyapunov exponent

$$\lambda(\varepsilon) = \lim_{t\uparrow\infty} t^{-1} \log|X_t| \tag{1.2}$$

and the rotation number, suitably defined in case $d = 2$. To analyze this system it is noted that the joint process (X_t, ξ_t) is Markovian on the product space $R^d \times M$ with infinitesimal generator

$$L = L_X = G + AX \cdot \nabla + \varepsilon(BX \cdot \nabla)F(\xi) \tag{1.3}$$

Here G is the generator of the noise (ξ_t); the second term is a "systematic derivative," making no reference to the noise process. The third term mixes the noise and state variables, inviting the term "noisy derivative" and is the source of the analytical challenge. This model is referred to as a "real-noise driven system."

It is well known that the noise process obeys a central limit theorem in the form

$$\delta \int_0^{t/\delta^2} F(\xi_s)ds \Rightarrow N(0, \sigma^2 t) \qquad (\delta \downarrow 0) \tag{1.4}$$

a normal law with mean zero and variance $\sigma^2 t$, where the variance parameter $\sigma^2 = -2\langle G^{-1}F, F\rangle$ and the inner product is computed in terms of the invariant measure of the process (ξ_t). Therefore one may attempt to analyze the asymptotics of (X_t, ξ_t) by studying a related system driven by *white noise*

$$dY_t = AY_t dt + \varepsilon BY_t \circ dw_t \tag{1.5}$$

where (w_t) is a Wiener process with mean zero and variance $\sigma^2 t$. The diffusion process (Y_t) is Markovian on R^d and has infinitesimal generator

$$L_Y = AY \cdot \nabla + \frac{1}{2}\varepsilon^2(BY \cdot \nabla)^2. \tag{1.6}$$

One may conjecture that the asymptotic behavior of the Lyapunov exponent for the real-noise driven process (X_t) is equivalent to that for the diffusion process (Y_t), at least to the first approximation. It is our purpose to carry out the details of this plan in cases of interest.

2. Nilpotent Systems.

In a previous paper [5] we investigated the Lyapunov exponent for white noise systems with a nilpotent deterministic part: $A^d = 0$, A^{d-1} non-zero and B generic. This includes the free particle perturbed by multiplicative white noise as well as other models of physical interest. Recently these results were extended to systems driven by "telegraphic noise," where $M = \{-1, 1\}$, by Arnold and Kloeden [7]. Now we can show that these results can be extended to the general real-noise driven system. We have the following theorem.

THEOREM 2.1. *Let $\lambda(\varepsilon)$, $r(\varepsilon)$ be the Lyapunov exponent and rotation number of the 2×2 system $X_t' = [A + \varepsilon BF(\xi_t)]X_t$ where $A^2 = 0$, A non-zero and (ξ_t) is a finite-state ergodic Markov process. Suppose that $\langle Be^{\perp}, e\rangle > 0$ where $Ae = 0$, e non-zero, $\langle e, e^{\perp}\rangle = 0$. When $\varepsilon \downarrow 0$ we have*

$$(2.1) \qquad \lambda(\varepsilon) \sim C_1 \varepsilon^{2/3} \qquad r(\varepsilon) \sim C_2 \varepsilon^{2/3}$$

for positive constants C_1, C_2. These expansions are precisely the same as for the Lyapunov exponent and rotation number of the associated diffusion process (Y_t).

We compute the Lyapunov exponent by the "adjoint method." This consists in writing the generator in terms of a system of polar coordinates (ρ, φ) and setting $Q(\xi, \varphi) = L(\rho)$. The angular process (ξ_t, φ_t) is ergodic with stationary measure $N(d\xi \times d\varphi)$ and the usual formula for the Lyapunov exponent is $\lambda(\varepsilon) = \int_{M \times S^{d-1}} Q(\xi, \varphi) N(d\xi \times d\varphi)$. In the present case, $\lambda(\varepsilon)$ may be characterized as the unique number λ for which there exists a function $f(\varphi, \xi)$ solving the equation $Lf = Q - \lambda$. Indeed, integrating both sides against $N(d\xi \times d\varphi)$ shows that $\lambda = \lambda(\varepsilon)$. Since this equation may be difficult to solve exactly, we may obtain asymptotic approximations by replacing L by a suitable approximate generator, or equivalently to find a function f_ε and a number λ_ε such that $Lf_\varepsilon = Q - \lambda_\varepsilon + O(R_\varepsilon)$ for a suitable remainder term R_ε. Integrating this equation against $N(d\xi \times d\varphi)$ produces the asymptotic statement $\lambda_\varepsilon = \lambda(\varepsilon) + O(R_\varepsilon)$. It remains to find the approximations f_ε, λ_ε, R_ε.

To do this we apply a method of "homogenization." We write the noisy part of the generator in the form $G + \delta V$ and show that this is approximated by a diffusion operator in the following sense: there exists a second order operator $L_o : C^\infty(R^2) \to C^\infty(R^2)$ and operators $L_i : C^\infty(R^2) \to C^\infty(R^2 \times M)$ $(i = 1, 2)$ such that for each $f \in C^\infty(R^2)$ we have

$$(2.2) \qquad (G + \delta V)(f + \delta L_1 f + \delta^2 L_2 f) = \delta^2 (L_o f) + O(\delta^3) \qquad \delta \downarrow 0)$$

This allows us to reduce to the case of a white-noise drive system for which we know the asymptotics of the Lyapunov exponent and rotation number. The details appear in sections 4 and 5.

3. The Harmonic Oscillator.

We now compare the small-noise behavior of the stochastic harmonic oscillator for the cases of real noise and white noise. The real noise system is defined by $X_t' = AX_t dt + \varepsilon BX_t F(\xi_t)$ where $A = \begin{pmatrix} 0 & 1 \\ -\gamma & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\gamma > 0$. We assume specifically that (ξ_t) is a finite-state Markov process with self-adjoint generator (reversible case) and invariant measure $\nu(d\xi)$. We take a system of polar coordinates (ρ, φ) with $x_1\sqrt{\gamma} = e^{\rho}\cos\varphi$, $x_2 = e^{\rho}\sin\varphi$. The stochastic equations take the form

$$
\begin{aligned}
\varphi_t' &= -\sqrt{\gamma} + \varepsilon[F(\xi_t)/\sqrt{\gamma}]\cos^2\varphi_t \\
\rho_t' &= \varepsilon[F(\xi_t)/\sqrt{\gamma}]\sin\varphi_t\cos\varphi_t
\end{aligned}
\tag{3.1}
$$

The infinitesimal generator of the joint motion of $(\xi_t, \varphi_t, \rho_t)$ is

$$
L_X = G - \sqrt{\gamma}\frac{\partial}{\partial\varphi} + \varepsilon[F(\xi)/\sqrt{\gamma}]\left(\cos^2\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\right).
$$

The white noise system is defined by the Stratonovich equation

$$
dY_t = AY_t dt + \varepsilon BY_t \circ dw_t \tag{3.2}
$$

where w_t is a Wiener process with mean zero and variance $\sigma^2 t$. Its generator is given by

$$
L_Y = -\sqrt{\gamma}\frac{\partial}{\partial\varphi} + \frac{1}{2}\varepsilon^2\sigma^2(\cos^2\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\cos\varphi\frac{\partial}{\partial\rho})^2
$$

where we make the identification $\sigma^2 = -2\langle G^{-1}F, F\rangle$.

To obtain the asymptotic form of the Lyapunov exponent of the real-noise driven system we look for an approximate solution (f, λ) of the equation $L_X f = Q - \lambda$ where $Q(\xi, \varphi) = L\rho = \varepsilon[F(\xi)/\sqrt{\gamma}]\sin\varphi\cos\varphi$. This is sought in the form

$$
\begin{aligned}
f &= f_0 + \varepsilon f_1 + \varepsilon^2 f_2 \\
\lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2
\end{aligned}
\tag{3.3}
$$

leading to the conditions

$$
\begin{aligned}
(G - \sqrt{\gamma}\frac{\partial}{\partial\varphi})f_0 &= 0 \\
(G - \sqrt{\gamma}\frac{\partial}{\partial\varphi})f_1 + [F(\xi)/\sqrt{\gamma}]\cos^2\varphi f_0' &= [F(\xi)/\sqrt{\gamma}]\sin\varphi\cos\varphi - \lambda_1 \\
(G - \sqrt{\gamma}\frac{\partial}{\partial\varphi})f_2 + [F(\xi)/\sqrt{\gamma}]\cos^2\varphi f_1' &= -\lambda_2.
\end{aligned}
$$

This leads to $f_0 = 0$, $\lambda_1 = 0$, $f_1 = (G - \frac{\partial}{\partial\varphi})^{-1}F(\xi)\sin\varphi\cos\varphi/\sqrt{\gamma}$ which may be solved in terms of eigenfunctions ψ_k where $G\psi_k = -\mu_k\psi_k$ for $k = 1, \ldots, N$ with $\psi_1 = 1$, $\mu_1 = 0$ and $\mu_k > 0$ for $k > 1$ by writing $F(\xi)\sin\varphi\cos\varphi = \frac{1}{2}\sum_{k>1}(F, \psi_k)\psi_k \sin 2\varphi$, leading to

$$f_1 = (1/\sqrt{\gamma})\sum_{k>1}\left[\frac{1}{2}\mu_k \sin 2\varphi + \sqrt{\gamma}\cos 2\varphi\right](F, \psi_k)\psi_k/(\mu_k^2 + 4\gamma).$$

Averaging the f_2 equation with respect to the normalized measure $d\varphi\nu(d\xi)$ and integrating by parts leads to

$$\begin{aligned} -\lambda_2 &= (1/\sqrt{\gamma})\int F(\xi)\cos^2\varphi f_1' d\varphi\nu(d\xi) \\ &= -(2/\sqrt{\gamma})\int F(\xi)\sin\varphi\cos\varphi f_1 d\varphi\nu(d\xi). \end{aligned}$$

From the above spectral representations, we can compute the inner product of f_1 and $F\sin 2\varphi$ to obtain

$$\int f_1 F(\xi)\sin\varphi\cos\varphi d\varphi\nu(d\xi) = \sum_{k>1}\mu_k(F, \psi_k)^2\langle\sin^2 2\varphi\rangle/4(\mu_k^2 + 4\gamma)$$

with the result ([4], [6])

$$\lambda_2^{\text{real}} = (1/4\gamma)\sum_{k>1}\mu_k(F, \psi_k)^2/(\mu_k^2 + 4\gamma). \tag{3.4}$$

To obtain the asymptotic form of the Lyapunov exponent of the white-noise driven system we look for an approximate solution (f, λ) of the equation $L_Y f = Q - \lambda$ where $Q(\xi, \varphi) = L_Y\rho = (\varepsilon^2/2\gamma)\cos^2\varphi\cos 2\varphi$. This is sought in the form

$$\begin{aligned} f &= f_0 + \varepsilon^2 f_2 \\ \lambda &= \lambda_0 + \varepsilon^2\lambda_2 \end{aligned}$$

leading to the conditions

$$\begin{gathered} -\sqrt{\gamma}f_0' = 0 \\ \sqrt{\gamma}f_2 + \frac{1}{2}\sigma(\cos\varphi\frac{\partial}{\partial\varphi})^2 f_0 = (\sigma^2/2\gamma)\cos^2\varphi\cos 2\varphi - \lambda_2 \end{gathered}$$

leading to the choices $f_0 = 0$ and the well known result (replacing $d\varphi$ by $d\varphi/2\pi$)

$$\begin{aligned} \lambda_2^{\text{white}} &= (\sigma^2/2\gamma)(2\pi)^{-1}\int_{-\pi}^{\pi}\cos^2\varphi\cos 2\varphi d\varphi \\ &= (\sigma^2/8\gamma). \end{aligned} \tag{3.5}$$

If we make the identification $\sigma^2 = -2\langle G^{-1}F, F\rangle$, then this may be written as

$$\lambda_2^{\text{white}} = (1/4\gamma)\sum_{k>1}(F, \psi_k)^2/\mu_k. \tag{3.6}$$

These computations are summarized as

PROPOSITION 3.1. *We always have the inequality* $\lambda_2^{\text{real}} < \lambda_2^{\text{white}}$.

To resolve this apparent discrepancy, it suffices to consider a parametrized family of real-noise processes $\int_0^t \delta^{-1} F(\xi_{s\delta^{-2}})ds$. When $\delta \downarrow 0$ these converge to a Wiener process with mean zero and variance $\sigma^2 = -2\langle G^{-1}F, F\rangle$. The stochastic equation has the form

$$X_t' = AX_t + \varepsilon[\delta^{-1}F(\xi_{t\delta^{-2}})]BX_t$$

with infinitesimal generator

$$L_X = AX \cdot \nabla + \delta^{-2}G + \varepsilon\delta^{-1}F(\xi)BX \cdot \nabla.$$

To obtain the corresponding form of the Lyapunov exponent, it suffices to substitute above. with μ_k replaced by $\mu_k\delta^{-2}$ and F replaced by $F\delta^{-1}$. Thus

$$\begin{aligned}\lambda_2^{\text{real}}(\delta) &= (1/4\gamma)\delta^{-2}\sum_{k>1}(\mu_k\delta^{-2})(F,\psi_k)^2/[(\mu_k\delta^{-2})^2 + 4\gamma] \\ &= (1/4\gamma)\sum_{k>1}\mu_k(F,\psi_k)^2/[\mu_k^2 + 4\gamma\delta^4].\end{aligned}$$

When $\delta \downarrow 0$ we have

PROPOSITION 3.2. $\lim_{\delta\downarrow 0}\lambda_2^{\text{real}}(\delta) = (1/4\gamma)\sum_{k>1}\langle F,\psi_k\rangle^2/\mu_k = \lambda_2^{\text{white}}$.

Thus we retrieve the white noise result in the CLT limit.

4. Proof of Theorem 2.1 (special case).

We are given an ergodic Markov process $\{\xi(t)\}_{t\geq 0}$ on a compact state space M; the infinitesimal generator is denoted G and the invariant measure ν—thus $G^*\nu = 0$ and $G1 = 0$. We further assume that the Fredholm alternative is satisfied for the simple eigenvalue zero, i.e., the inhomogeneous equation $Gf = g$ has a solution provided that $\int_M g(\xi)\nu(d\xi) = 0$; the solution is uniquely determined by requiring $\int_M f(\xi)\nu(d\xi) = 0$. This condition is satisfied for a finite-state Markov process or for Brownian motion on a compact manifold, for example.

Let there be given a function $F(\xi)$ with mean value zero, i.e., $\int_M F(\xi)\nu(d\xi) = 0$. Let $(x(t), x'(t)) = (x_1(t), x_2(t))$ be the solution of the second-order system $x''(t) = \varepsilon x(t)F(\xi(t))$ with the initial conditions $x(0) = x_1$, $x'(0) = x_2$. This is a Markov process on the product space $R^2 \times M$ with the infinitesimal generator

$$L = G + x_2\frac{\partial}{\partial x_1} + \varepsilon F(\xi)x_1\frac{\partial}{\partial x_2}. \tag{4.1}$$

The top Lyapunov exponent is defined by

$$\lambda(\varepsilon) = \lim_{t\uparrow\infty} t^{-1}\log\sqrt{x_1(t)^2 + x_2(t)^2}. \tag{4.2}$$

This is invariant under linear change of coordinates in (x_1, x_2) space, in particular the scaling transformation $(x_1, x_2) \to (x_1, Cx_2)$.

We introduce a system of "polar coordinates" by

$$x_1 = e^{\rho}\cos\varphi, \quad x_2 = Ce^{\rho}\sin\varphi. \tag{4.3}$$

We make the identification $x(t) = x_1(t)$, $x'(t) = x_2(t)$ and consider the joint process $(\xi(t), \rho(t), \varphi(t))_{t\geq 0}$. After a short calculation we find that

$$\begin{aligned} \varphi'(t) &= -C\sin^2\varphi(t) + (\varepsilon/C)\cos^2\varphi(t)F(\xi(t)) \\ \rho'(t) &= C\sin\varphi(t)\cos\varphi(t) + (\varepsilon/C)\sin\varphi(t)\cos\varphi(t)F(\xi(t)). \end{aligned} \tag{4.4}$$

The joint process $(\xi(t), \varphi(t), \rho(t))_{t\geq 0}$ is a Markov process on the space $M \times R \times R$ with the infinitesimal generator

$$\begin{aligned} L &= G + \varepsilon F(\xi)x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1} \\ &= G + (-C\sin^2\varphi + (\varepsilon/C)\cos^2\varphi F(\xi))\frac{\partial}{\partial\varphi} \\ &\quad + \sin\varphi\cos\varphi(C + (\varepsilon/C)F(\xi))\frac{\partial}{\partial\rho} \end{aligned} \tag{4.5}$$

(in fact the first two components already form a Markov process, but we shall need the full generator in what follows).

We write the generator in the form $L = G + \delta V + D$, where

$$\begin{aligned} V &= F(\xi)\left(\sin\varphi\cos\varphi\frac{\partial}{\partial\rho} + \cos^2\varphi\frac{\partial}{\partial\varphi}\right) \\ D &= C\sin\varphi\cos\varphi\frac{\partial}{\partial\rho} - C\sin^2\varphi\frac{\partial}{\partial\varphi} \end{aligned}$$

and $\delta = \varepsilon/C$. We refer to G as the *noise generator*, V as the *noisy derivatives* and D as the *systematic derivatives*. We also note, for further reference, the function $Q(\varphi, \xi)$ defined as $L\rho(\varphi, \xi)$ is computed as

$$Q(\varphi, \xi) = C\sin\varphi + (\varepsilon/C)F(\xi)\sin\varphi\cos\varphi. \tag{4.6}$$

In previous approaches to stochastic Lyapunov stability, one solves approximately the equation $Lf = Q - \lambda$ for suitable $f = f(\varphi, \xi)$ and $\lambda \in R$. In the present case this is not directly possible, because of the presence of the noise variables $F(\xi)$ which intervene both in the generator and in $Q(\varphi, \xi)$. Therefore we apply a process of "homogenization" to replace the operator $G + \delta V$ by a suitable diffusion operator in the (φ, ρ) space and ultimately replace $Q(\varphi, \xi)$ by a function $Q(\varphi)$ related to a suitable diffusion process.

PROPOSITION 4.1. *There exists a second-order differential operator L_0 in the (φ,ρ) variables with the following property: for any $f \in C^\infty(R\times R)$ there exist correctors $f_i \in C^\infty(R\times R\times M)$ $(i=1,2)$ such that*

$$
\begin{aligned}
&[G+\delta V](f+\delta f_1+\delta^2 f_2)\\
(4.7)\qquad &=: \left[G+\left(\delta F(\xi)(\cos^2\varphi\frac{\partial}{\partial\varphi}+\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\right)\right](f+\delta f_1+\delta^2 f_2)\\
&=\delta^2(L_o f)(\varphi,\rho)+O(\delta^3)\qquad(\delta\downarrow 0)
\end{aligned}
$$

where $L_o f(\rho,\varphi)=\frac{1}{2}\sigma^2\left(\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}+\cos\varphi\frac{\partial}{\partial\varphi}\right)^2 f$ *and* $\sigma^2=-2\langle G^{-1}F,F\rangle>0$. *The term $O(\delta^3)$ is estimated in terms of the C^3 norm of f.*

PROOF: We choose the correctors in order to cancel the term of order δ and to render the coefficient of δ^2 independent of ξ. This requires that we have the equations

$$(4.8)\qquad Gf_1-\cos^2\varphi F(\xi)\frac{\partial f}{\partial\varphi}-\sin\varphi\cos\varphi F(\xi)\frac{\partial f}{\partial\rho}=0$$

$$(4.9)\qquad Gf_2-\cos^2\varphi F(\xi)\frac{\partial f_1}{\partial\varphi}-\sin\varphi\cos\varphi F(\xi)\frac{\partial f_1}{\partial\rho}=\text{ function of }(\varphi,\rho).$$

The first of these is satisfied by taking

$$f_1=-H(\xi)\left[\cos^2\varphi\frac{\partial f}{\partial\varphi}+\sin\varphi\cos\varphi\frac{\partial f}{\partial\rho}\right]$$

where $H(\xi)$ is the solution of $GH=-F$, normalized so that $\int_M H(\xi)\nu(d\xi)=0$. With this choice of f_1 we substitute in the equation (4.9) for f_2 and average with respect to $\nu(d\xi)$. The Gf_2 term drops out and the right side of the equation is found to be

$$(4.10)\qquad \frac{1}{2}\sigma^2\left(\cos^2\varphi\frac{\partial}{\partial\varphi}+\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\right)^2 f=:L_o f$$

where $\sigma^2=2\int_M F(\xi)H(\xi)\nu(d\xi)$.

Finally f_2 is determined by solving the indicated equation (4.9) subject to the normalization $\int_M f_2(\xi,\varphi,\rho)\nu(d\xi)=0$. This is possible, since Gf_2 has been arranged to be perpendicular to the null space of G^*, completing the proof.

If $\varepsilon/C\downarrow 0$ we may restrict attention to the "approximate generator" $(\varepsilon/C)^2L_o f-C\sin^2\varphi\frac{\partial}{\partial\varphi}+C\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}$. In order for the terms to balance we are led to the equation $(\varepsilon/C)^2=C$ or $C=\varepsilon^{2/3}$ as we had in the diffusion case [5]. More precisely, we consider the white noise system $dx=x_2dt$, $dx_2=\varepsilon x_1\circ dw$ where $\{w(t):t\geq 0\}$ is a Wiener process with mean zero and variance parameter σ^2t. For this system the infinitesimal generator is $\tilde{L}_\varepsilon=x_2\frac{\partial}{\partial x_1}+\varepsilon^2\frac{\sigma^2}{2}\left(x_1\frac{\partial}{\partial x_2}\right)^2$; if we take the polar coordinate system $x_1=e^\rho\cos\varphi$, $x_2=Ce^\rho\sin\varphi$, we obtain the angular equation

$$(4.11)\qquad d\varphi=-C\sin\varphi\cos\varphi+(\sigma\varepsilon/C)^2\cos^2\varphi\circ dw$$

with the Q function ($Q_\varepsilon = \tilde{L}_\varepsilon \rho$)

$$Q_\varepsilon(\varphi) = -C \sin\varphi \cos\varphi + (\sigma\varepsilon/C)^2 \cos^2\varphi \cos 2\varphi. \tag{4.12}$$

From our previous work [5], we know that if we choose $C = \varepsilon^{2/3}$, then $\tilde{L}_\varepsilon = \varepsilon^{2/3}\tilde{L}_1$, where $\tilde{L}_1$ is hypoelliptic with invariant measure μ and the Lyapunov exponent is $\lambda_\varepsilon = \int_{-\pi}^{\pi} Q_1(\varphi)\mu(d\varphi) > 0$.

In order to find the Lyapunov exponent of the real-noise driven system, it suffices to find $f(\varphi,\xi)$ and $\lambda^\sim$ such that $Lf(\varphi,\xi) = Q(\varphi,\xi) - \lambda^\sim + O(\varepsilon)$. To do this, we proceed in three steps.

Step 1. Let $\rho_1(\varphi,\xi)$, $\rho_2(\varphi,\xi)$ be the correctors such that

$$(G + \delta V)(\rho + \delta\rho_1 + \delta^2\rho_2) = \delta^2 L_o\rho + O(\delta^3) \tag{4.13}$$

In our previous paper [5] we showed that the operator $-\sin^2\varphi\frac{\partial}{\partial\varphi} + L_0$ on $[-\pi,\pi]$ has an invariant measure μ and satisfies the Fredholm alternative for the simple eigenvalue zero and that $\lambda_1 =: \int_{-\pi}^{\pi} Q(\varphi)\mu(d\varphi) > 0$, where $Q(\varphi) =: \sin\varphi\cos\varphi + L_o\rho = \sin\varphi\cos\varphi + \sigma^2\cos^2\varphi\cos 2\varphi$.

Step 2. Let $h = h(\varphi)$ be the solution of the equation

$$\left(-\sin^2\varphi\frac{\partial}{\partial\varphi} + L_o\right)h = Q(\varphi) - \lambda_1 \tag{4.14}$$

normalized so that $\int_{-\pi}^{\pi} h(\varphi)\mu(d\varphi) = 0$.

Step 3. Let $h_1(\varphi,\xi)$, $h_2(\varphi,\xi)$ be the correctors defined above for the function h, i.e., $(G+\delta V)(h + \delta h_1 + \delta^2 h_2) = \delta^2 L_o h + O(\delta^3)$.

PROPOSITION 4.2. *With the above notations, we let*

$$f(\varphi,\xi) = h(\varphi) + \delta(h_1 - \rho_1)(\varphi,\xi) + \delta^2(h_2 - \rho_2)(\varphi,\xi) \tag{4.15}$$

where $\delta = \varepsilon/C$, $C = \varepsilon^{2/3}$. Then $Lf(\varphi,\xi) = Q(\varphi,\xi) - \varepsilon^{2/3}\lambda_1 + O(\varepsilon)$ in particular the Lyapunov exponent $\lambda(\varepsilon) = \varepsilon^{2/3}\lambda_1 + O(\varepsilon)$, $\varepsilon \downarrow 0$.

PROOF: Recall that $L = G + \delta V + D$ where G is the noise generator, V contains the noisy derivatives and D contains the systematic derivatives. From step 1, we have

$$\begin{aligned}(G+\delta V)(\rho + \delta\rho_1 + \delta^2\rho_2) &= \delta^2\sigma^2\cos^2\varphi\cos 2\varphi + O(\varepsilon)\\ L(\rho + \delta\rho_1 + \delta^2\rho_2) &= C\sin\varphi\cos\varphi + \delta^2\sigma^2\cos^2\varphi\cos 2\varphi + O(\varepsilon)\\ &= \varepsilon^{2/3}Q(\varphi) + O(\varepsilon)\end{aligned}$$

$$\begin{aligned}(G+\delta V)(h + \delta h_1 + \delta^2 h_2) &= \delta^2 L_o h + O(\varepsilon)\\ L(h + \delta h_1 + \delta^2 h_2) &= -C\sin^2\varphi\, h'(\varphi) + \delta^2 L_o h + O(\varepsilon)\\ &= \varepsilon^{2/3}(Q(\varphi) - \lambda_1) + O(\varepsilon).\end{aligned}$$

Subtracting these two gives

$$L(h-\rho+\delta(h_1-\rho_1)+\delta^2(h_2-\rho_2))=-\varepsilon^{2/3}\lambda_1+O(\varepsilon).$$

Adding $Q(\varphi,\xi)=L\rho$ to both sides gives the desired result. To complete the proof. we may argue directly in terms of martingales: writing the above equation as $L\tilde{f}=-\varepsilon^{2/3}\lambda_1+O(\varepsilon)$, we have that $M_t=:\tilde{f}(\rho(t),\varphi(t),\xi(t))-\int_0^t L\tilde{f}(\rho(s),\varphi(s),\xi(s)ds$ is a martingale. But $M_t=o(t)$ when $t\uparrow\infty$; therefore we may divide by t and take the limit obtaining $\lim_{t\uparrow\infty}\rho(t)/t=-\lambda_1\varepsilon^{2/3}+O(\varepsilon)$. We have the following theorem.

THEOREM 2.1(A). *The top Lyapunov exponent satisfies*

$$\lambda(\varepsilon)=\lim_{t\uparrow\infty}t^{-1}\rho(t)=\varepsilon^{2/3}\langle\sin\varphi\cos\varphi+\sigma^2\cos^2\varphi\cos 2\varphi\rangle_\mu+O(\varepsilon)\qquad(\varepsilon\downarrow 0).$$

By the same method we can compute the rotation number, defined as $r(\varepsilon)=\lim_{t\uparrow\infty}t^{-1}\varphi(t)$. To do this we verify the following:

PROPOSITION 4.3. *Let $k(\varphi)$ be the solution of*

$$-\sin^2\varphi k'(\varphi)+L_o k=L_o\varphi-\sin^2\varphi-\langle L_o\varphi\rangle_\mu+\langle\sin^2\varphi\rangle_\mu.$$

Let $k_1(\varphi)$, $k_2(\varphi)$ be the correctors defined above for $f=\varphi+k(\varphi)$. Then $L(\varphi+k(\varphi)+\varepsilon^{1/3}k_1(\varphi)+\varepsilon^{2/3}k_2(\varphi))=\varepsilon^{2/3}\langle L_o\varphi+\sin^2\varphi\rangle_\mu+O(\varepsilon)$.

Noting that $k(\varphi)$ and the correctors φ_1, φ_2, k_1, k_2 are periodic functions we have the following result:

THEOREM 2.1(B). *The rotation number is computed as*

$$r(\varepsilon)=\lim_{t\uparrow\infty}\varphi(t)/t=\varepsilon^{2/3}\langle\sin\varphi\cos\varphi+\sigma^2\sin^2\varphi\rangle_\mu+O(\varepsilon)\qquad(\varepsilon\downarrow 0).$$

5. Proof of Theorem 2.1 (general case).

We now generalize the set-up of the previous section to the stochastic system

$$x'(t)=[A+\varepsilon F(\xi(t))B]x(t)\tag{5.1}$$

where $x(t)=(x_1(t),x_2(t))$ and $A\neq 0$ is a 2×2 matrix with $A^2=0$. Without loss of generality we may take a basis in which $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ so that the system has the form

$$\begin{aligned}x_1'(t)&=x_2(t)+\varepsilon F(\xi(t))(b_{11}x_1(t)+b_{12}x_2(t))\\x_2'(t)&=\varepsilon F(\xi(t))(b_{21}x_1(t)+b_{22}x_2(t)).\end{aligned}\tag{5.2}$$

Taking a system of polar coordinates $x_1 = e^\rho \cos\varphi$, $x_2 = Ce^\rho \sin\varphi$ we find that

$$\begin{aligned}\varphi' &= -C\sin^2\varphi - \varepsilon\sin\varphi(b_{11}\cos\varphi + b_{12}C\sin\varphi)F(\xi)\\ &\quad + (\varepsilon/C)\cos\varphi(b_{21}\cos\varphi + b_{22}C\sin\varphi)F(\xi)\\ \rho' &= C\sin\varphi\cos\varphi + \varepsilon\cos\varphi(b_{11}\cos\varphi + b_{12}C\sin\varphi)F(\xi)\\ &\quad + (\varepsilon/C)\sin\varphi(b_{21}\cos\varphi + b_{22}C\sin\varphi(F(\xi).\end{aligned}$$

Making the choice $C = \varepsilon^{2/3}$, the generator has the form

$$\begin{aligned}L = -\varepsilon^{2/3}&\left\{\sin^2\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\right\}\\ &+ G + \varepsilon^{1/3}V_1 + \varepsilon V_2 + \varepsilon^{5/3}V_3\end{aligned}$$

where

$$\begin{aligned}V_1 &= b_{21}\cos^2\varphi F(\xi)\frac{\partial}{\partial\varphi} + b_{21}F(\xi)\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\\ V_2 &= (b_{22}-b_{11})\sin\varphi\cos\varphi F(\xi)\frac{\partial}{\partial\varphi} + (b_{11}\cos^2\varphi + b_{22}\sin^2\varphi)F(\xi)\frac{\partial}{\partial\rho}\\ V_3 &= -b_{12}\sin^2\varphi\frac{\partial}{\partial\varphi} + b_{12}\sin\varphi\cos\varphi\frac{\partial}{\partial\rho}.\end{aligned}$$

This has the same structure as in the case of white noise. If the coefficient b_{21} is non-zero. we may apply the homogenization procedure described above to obtain the approximate generator as

$$\varepsilon^{2/3}b_{21}^2 L_o f - \varepsilon^{2/3}\left\{\sin^2\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\cos\varphi\frac{\partial}{\partial\rho}\right\} + O(\varepsilon).$$

Applying the adjoint method we again obtain the result in the form as stated in Theorem 2.1.

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